K12 SCHOOLS
YOUR EDUCATION PORTAL

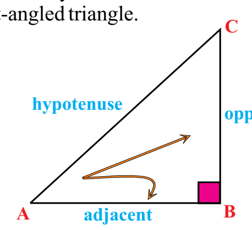
MATHEMATICS
TRIGONOMETRY

INTRODUCTION

What is Trigonometry ? Within our context it can be safely said that Trigonometry is a study of Similar right-angled triangles either in isolation, or as triangles formed from rotation in the Cartesian Plane. We relate measurements of angles to the measurement of corresponding sides by the use of the functions sin θ , cos θ and tan θ. These functions define specific division of sides in a right-angled triangle.

DEFINITIONS IN A RIGHT-ANGLED TRIANGLE

In ΔABC with reference to Â or θ ,
we say that BC is the **opposite** side ,
AB is the **adjacent** side , and,
AC is the **hypotenuse**.



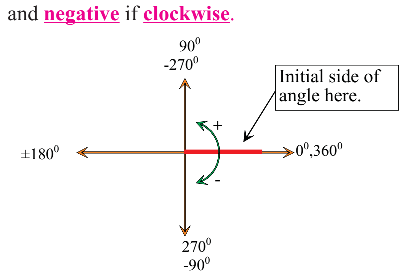
With reference to θ the ratios are defined as follows :

$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{O}{H} = \sin\theta \qquad \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A}{H} = \cos\theta \qquad \frac{\text{opposite}}{\text{adjacent}} = \frac{O}{A} = \tan\theta$$

SOH CAH TOA is a famous way of remembering this

TRIGONOMETRY IN THE CARTESIAN SYSTEM

Of interest to us is the measure of rotation that starts at 0^0. Angle measure is **positive** if it is **anti-clockwise** and **negative** if **clockwise**.

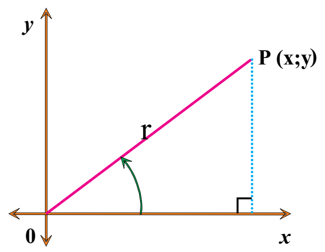


Note that the angle is formed between the initial side of the ray at 0^0 and its terminal side. In all cases our angles are in **standard position**, which means that the **initial side of the ray is at 0^0.**

When the ray rotates one of two things occur :
the terminal side ends on an axis. Angles are said to be **quadrantal.** These are $\{0^0;\pm90^0;\pm180^0;\pm270^0;\pm360^0\}$
the terminal side ends up in one of the four quadrants.

DEFINITIONS IN TERMS OF x ; y ; r and θ

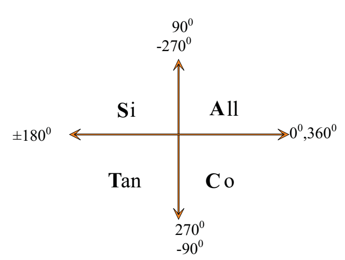
$\sin\theta = \frac{y}{r}$
$\cos\theta = \frac{x}{r}$
$\tan\theta = \frac{y}{x}$



To remember:
Shade **Y**our **R**ear ,
Cos **X R**ays ,
Tan **Y**our **X**terior.

THE SIGN DIAGRAM

The sign diagram enables us to see where the trigonometric ratios are **positive or negative**. We use it extensively to reduce from rotations in the 2^{nd} ,3^{rd} and 4^{th} quadrants to Angles in the 1^{st} quadrant and when solving equations.



To remember: All Spell Trigonometry Correctly

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

We have observed the following two **relationships** that exist between **trigonometric functions**.

1. $\tan\theta = \frac{\sin}{\cos}$
2. $\sin^2\theta + \cos^2\theta = 1$ which allows us to write $\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$

Basic Application Using Trigonometric Identities

Example 1. Prove $(\sin + \cos)^2 = 1 + 2\sin.\cos$

L.H.S = sin cos ^{2}
$\sin^2$ 2 sin cos $\cos^2$
2 sin cos $\sin^2$ $\cos^2$
1 2 sin cos
R.H.S

Here we need to expand the bracket

Grouping the square identity

Example 2. Prove $\sin^2x + \tan x.\frac{\cos x}{\sin x} + \cos^2x = 2$

L.H.S= $\sin^2x$ $\tan x.\frac{\cos x}{\sin x}$ $\cos^2 x$
$\sin^2x$ $\cos^2 x$ $\frac{\sin x}{\cos x}.\frac{\cos x}{\sin x}$
1 1
2
R.H.S

tan x was written in terms of sin x and cos x

REDUCTION FORMULA

Reduction formula allows us to convert from an **angle of rotation** which is **more than 90°** to an angle in the **first quadrant**. In fact, that is the driving force behind simplification.

To be able to reduce to an acute angle note that :

- **anti-clockwise rotation is positive** and **a clockwise rotation is negative.**
- the initial side of a rotating ray is at 0^0, the positive side of the x-axis.

When confronted with a rotation that has to be written as a positive acute angle , ask yourself these three questions :

a. In which **Quadrant** is the ray after rotation Δ
b. What is the **Sign** of the function there Δ
c. **Ratio changes** for (90° θ) reduction to the co-function.

sin ⟷ cos

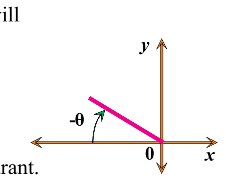
To remember use: Quick Simple Reasoning.

Example 1 Rotation of 180^0 - θ

sin (180^0-θ)

Visualise where 180^0-θ will take the ray.

180^0 is on the negative x-axis.
- θ means we go back an acute angle.
The ray is in the 2^{nd} quadrant.



In the 2^{nd} quadrant only the sine function is positive.
Our answer is written as:
sin (180^0-θ) = sin θ

In the 2^{nd} quadrant both the cosine and tangent functions are negative.
Therefore cos (180^0-θ) = - cos θ and tan (180^0-θ)= - tanθ

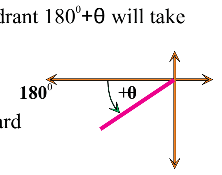
T

Example 2 Rotation of 180^0 + θ

an (180^0+θ)

Visualise in which quadrant 180^0+θ will take the ray.

180^0 is on the negative x-axis.
+ θ means we go forward a small acute angle.
The ray is in the 3^{rd} quadrant.



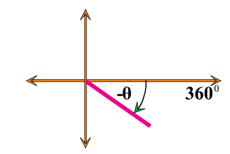
In the 3^{rd} quadrant only the tangent function is positive.
Our answer is written as
tan (180^0+θ) = tan θ

In the 3^{rd} quadrant both the cosine and sine functions are negative.
Therefore cos (180^0+θ) = -cos θ and sin (180^0+θ)=- sinθ

Example 3 Rotation of 360^0 - θ

cos (360^0- θ)

360^0 is on the positive X-axis.
- θ means we go back a small acute angle.
The ray is in the 4^{th} quadrant.



In the 4^{th} quadrant only the cosine function is positive. Our answer is written as : **cos (360^0-θ) = cos θ**

In the 4^{th} quadrant both the sine and tangent functions are negative.
Therefore sin(360^0-θ) = -sin θ and tan (360^0-θ)=-tanθ

Application of Reduction Theory

Example 1 Simplify the following:

$$\tan 180^0 + \theta .\cos 540^0 + \theta \ \sin -\theta + \frac{\sin^2 90^0 + \theta}{\cos 90^0 + \theta}$$

tan . cos sin $\frac{\cos^2}{\sin}$ ← using reduction
$\frac{\sin}{\cos}$. cos sin $\frac{\cos^2}{\sin}$ ← Writing tan in terms of sin and cos
sin sin $\frac{\cos^2}{\sin}$
$\sin^2$ $\cos^2$
1

1. $\frac{\tan(180^0 - x)\sin(360^0 - x)\cos(90^0 - x)}{\sin(180^0 + x)\tan(180^0 + x)}$

$\frac{(\tan x)(\sin x)(\sin x)}{(\sin x)(\tan x)}$
sin x
cancelling common factors

2. $\frac{\tan(360^0 - x)\cos(90^0 - x)\sin(720^0 + x)}{\tan(180^0 + x)\sin(180^0 + x)\sin(180^0 - x)}$

$\frac{(\tan x)(\sin x)(\sin x)}{(\tan x)(\sin x)(\sin x)}$
1
cancelling common factors

3. $\frac{\tan(180^0 - x).\sin(90^0 + x)}{\sin(-x)} - \sin y.\cos(90^0 - y)$

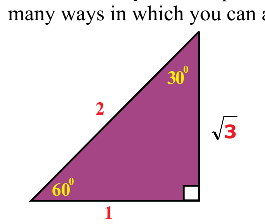
$\frac{(\tan x)(\cos x)}{(\sin x)}$ sin y.sin y
$\frac{(\frac{\sin x}{\cos x})(\cos x)}{(\sin x)}$ $\sin^2 y$
1 $\sin^2 y$
$\cos^2 y$

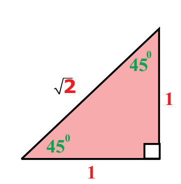
Writing tan as $\frac{\sin}{\cos}$

SPECIAL ANGLES

Function values of 30° ; 45° ; 60°

In this section you are expected to obtain the values of these special angles without calculators. There are many ways in which you can achieve this. We will use the following diagrams.





Example 1: sin 45°

The value of sin 45° = $\frac{1}{\sqrt{2}}$. We rationalize the denominator by multiplying the value by $\frac{\sqrt{2}}{\sqrt{2}}$ giving:

sin 45^0 $\frac{1}{\sqrt{2}}$ $\frac{\sqrt{2}}{\sqrt{2}}$
$\frac{\sqrt{2}}{2}$

Simplification Involving Reduction Theory

Simplify the following without calculators:

1. $\frac{\cos 120^0.\tan 135^0.\sin 150^0}{\sin 150^0.\cos 540^0}$

cos 180^0 60^0 tan 180^0 45^0
cos 540^0 360^0
$\frac{\cos 60^0 \quad \tan 45^0}{\cos 180^0}$ cancel sin 150°
$\frac{1}{2}$ 1
1
$\frac{1}{2}$

2. $\frac{-\sin 120^0.\cos 150^0 + \sin^2 135^0 - \cos 60^0}{\sin^3 135^0.\cos 135^0}$

sin 180^0 60^0 cos 180^0 30^0 $\sin^2$ 180^0 45^0 $\frac{1}{2}$
$\sin^3$ 180^0 45^0 cos 180^0 45^0

sin 60^0 cos 30^0 $\sin^2$ 45^0 $\frac{1}{2}$
$\sin^3 45^0$ cos 45^0

use a calculator for these computations

$\frac{\sqrt{3}}{2}$ $\frac{\sqrt{3}}{2}$ $\frac{1}{2}$ $\frac{1}{2}$
$\frac{1}{\sqrt{2}}^3$ $\frac{1}{\sqrt{2}}$

$\frac{3}{4}$
$\frac{1}{4}$
3.

COMPOUND AND DOUBLE ANGLES

In this aspect we will cover applications of compound angles. The formulae will be on your data sheet. You must be able to derive and use the following identities:

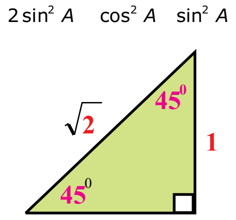
cos(A B) cos A cos B sin A sin B
cos(A B) cos A cos B sin A sin B
sin(A B) sin A cos B cos A sin B
sin(A B) sin A cos B cos A sin B
sin 2A 2 sin A cos A
cos 2A 2 $\cos^2$ A 1 1 2 $\sin^2$ A $\cos^2$ A $\sin^2$ A

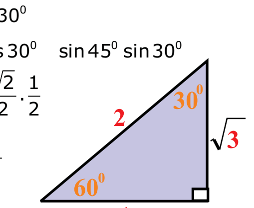
Simplifying Compound and Double Angle Forms

Combination of Special Angles

1. Cos75°

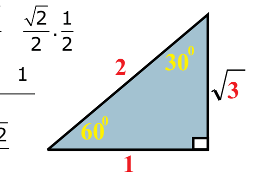
cos 75^0
cos 45^0 30^0
cos 45^0 cos 30^0 sin 45^0 sin 30^0
$\frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2}$ $\frac{\sqrt{2}}{2}.\frac{1}{2}$
$\frac{\sqrt{2}\ \sqrt{3}\ \ 1}{4}$
$\frac{\sqrt{6}\ \ \sqrt{2}}{4}$





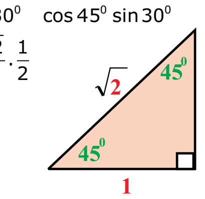
2. Cos15°

cos 45^0 30^0
cos 45^0 cos 30^0 sin 45^0 sin 30^0
$\frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2}$ $\frac{\sqrt{2}}{2}.\frac{1}{2}$
$\frac{\sqrt{2}\ \sqrt{3}\ \ 1}{4}$
$\frac{\sqrt{6}\ \ \sqrt{2}}{4}$



4. sin 15°

sin 45^0 30^0
sin 45^0 cos 30^0 cos 45^0 sin 30^0
$\frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2}$ $\frac{\sqrt{2}}{2}.\frac{1}{2}$
$\frac{\sqrt{2}\ \sqrt{3}\ \ 1}{4}$
$\frac{\sqrt{6}\ \ \sqrt{2}}{4}$



3. Show that $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$.

cos A cos 2 $\frac{A}{2}$
$\cos^2$ $\frac{A}{2}$ $\sin^2$ $\frac{A}{2}$

5 Simplify without a calculator: cos64° + sin 64°.tan32°

cos 64^0 sin 64^0 tan 32^0
cos 2 32^0 sin 2 32^0 tan 32^0
cos 2.32^0 2 sin 32^0 ~~cos 32^0~~ $\frac{\sin 32^0}{\text{~~cos 32°~~}}$
1 2 $\sin^2$ 32^0 2 $\sin^2$ 32^0
1

Proving identities using compound angle theory.

1. Prove that $\frac{2\sin^2x}{\tan x} = \sin 2x$.

L.H.S $\frac{2\sin^2 x}{\tan x}$
$\frac{2\sin^2 x}{\frac{\sin x}{\cos x}}$ ← writing tan x as sin x
$\frac{2\sin^2 x}{1}$ $\frac{\cos x}{\sin x}$ ← invert and multiply
2 sin x cos x
sin 2x
R.H.S

Proving using Double and Compound Angles

2. Prove that $\frac{\cos 2x + \sin^2x}{1 + \sin x} = 1 - \sin x$.

L.H.S $\frac{\cos^2 x \quad \sin^2 x \quad \sin^2 x}{1 \quad \sin x}$
$\frac{1 \quad \sin^2 x}{1 \quad \sin x}$ ← difference of 2 squares
$\frac{1 \quad \sin x \quad 1 \quad \sin x}{1 \quad \sin x}$
1 sin x
R.H.S

3. Prove that $\frac{\sin 2A + \cos A}{\cos 2A - \sin A - 1} = \frac{\sin(90^0 + A)}{\cos(90^0 + A)}$

L.H.S $\frac{2\sin A\cos A \quad \cos A}{1 \quad 2\sin^2 A \quad \sin A \quad 1}$
$\frac{\cos A \ \ 2\sin A \quad 1}{2\sin^2 A \quad \sin A}$
$\frac{\cos A \ \ 2\sin A \quad 1}{\sin A \ \ 2\sin A \quad 1}$
$\frac{\cos A}{\sin A}$

R.H.S $\frac{\sin\ 90^0 \quad A}{\cos\ 90^0 \quad A}$
$\frac{\cos A}{\sin A}$
$\frac{\cos A}{\sin A}$

L.H.S =

4. Prove $\frac{2\sin^2x - \sin 2x}{1 - 2\sin x.\cos x} = \frac{2\sin x}{\sin x - \cos x}$

L.H.S $\frac{2\sin^2 x \quad \sin 2x}{1 \quad 2\sin x\cos x}$ — Writing 1 as $\sin^2x+\cos^2x$
$\frac{2\sin^2 x \quad 2\sin x\cos x}{\sin^2 x \quad 2\sin x\cos x \quad \cos^2 x}$ ← Common factor
$\frac{2\sin x \quad \sin x \quad \cos x}{\sin x \quad \cos x^{\ 2}}$ — Factorisation of top and bottom
$\frac{2\sin x}{\sin x \quad \cos x}$
R.H.S

TRIGONOMETRIC EQUATIONS & GENERAL SOLUTIONS

SIN θ AND COS θ COMBINATIONS

1. sin = cos

sin cos
$\frac{\sin}{\cos}$ 1 ← Divide by cosθ
tan 1 ← For tan θ do one solution.
45^0 k.180^0, k

2. sin 3 = cos

sin3 cos ← For different angles use co-functions
sin3 sin 90^0

1st quadrant		2nd quadrant
3 90^0 k.360^0	or	3 180^0 90^0 k.360^0
4 90^0 k.360^0	or	3 90^0 k.360^0
22.5^0 k.90^0	or	2 90^0 k.360^0
22.5^0 k.90^0	or	45^0 k.180^0 , k

3. sin cos = 0.3

sin .cos 0,3
2 sin cos 0,6 ← Multiply by 2 to get in form of double angle
sin 2 0,6
positive acute angle 2nd F , sin , (0,6) 36,9^0

1st quadrant		2nd quadrant
2 36,9^0 k.360^0	or	2 180^0 36,9^0 k.360^0
18,4^0 k.180^0	or	2 143,13^0 k.360^0
18,4^0 k.180^0	or	71,6^0 k.180^0 , k

4. $\cos(3 - 10^0) = \sin(2 + 15^0)$

The cosine function is positive in the 1^{st} and 4^{th} quadrants.

cos 3 10^0 sin 2 15^0
cos 90^0 2 15^0
cos 75^0 2

1st quadrant		4th quadrant
3 10^0 75^0 2 k.360^0	or	3 10^0 360^0 75^0 2 k.360^0
5 85^0 k.360^0		3 10^0 285^0 2 k.360^0
17^0 k.72^0, k		295^0 k.360^0 , k

FACTORISATIONS

The following tutorial illustrates how the process of factorization is used to solve trigonometric equations.

1. 2 sin .cos - sin = 0

sin 2 cos 1 0
sin 0 *or* 2cos -1 0
p.a.a 0^0 cos $\frac{1}{2}$
0^0 k.180^0 p.a.a 60^0
1st Quadrant :
60^0 k.360^0
4th Quadrant :
360^0 60^0 k.360^0
300^0 k.360^0, k

2. $2\sin^2x = \sin x.\cos x + \cos^2x$

2 $\sin^2$ x sin x cos x $\cos^2$ x 0
2 sin x cos x sin x cos x 0

2 sin x cos x 0 *or* sin x cos x 0
2 sin x cos x sin x cos x
$\frac{\sin x}{\cos x}$ $\frac{1}{2}$ $\frac{\sin x}{\cos x}$ 1
tan x 0,5 tan x 1
p.a.a 26,6^0 p.a.a 45^0

We are not losing solutions by doing only one quadrant for tan θ. The second solution is obtained when k = 1. Try it …..

2nd Quadrant
x 180^0 26,6^0 k.180^0
x 153,4^0 k.180^0

1st Quadrant
x 45^0 k.180^0, k

COMPOUND AND SQUARE IDENTITIES

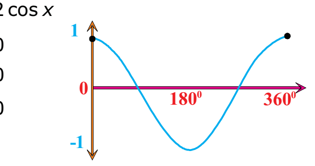
1. $2\cos^2 = 6\sin + 6$

$\cos^2$ 3 sin 3 ← Dividing by 2
1 $\sin^2$ 3 sin 3
$\sin^2$ 3 sin 1 3 0
$\sin^2$ 3 sin 2 0
sin 1 sin 2 0

sin 1 *or* sin 2
270^0 k.360^0, k no. solns

2. $2 - \sin^2x = 2\cos x$

2 $\sin^2$ x 2 cos x
2 1 $\cos^2$ x 2 cos x
1 $\cos^2$ x 2 cos x 0
$\cos^2$ x 2 cos x 1 0
cos x 1 ^{2} 0



cos x 1
x 0^0 k.360^0, k

3. $\cos 2\theta + 3\sin^2\theta - 4\sin\theta + 2 = 0$

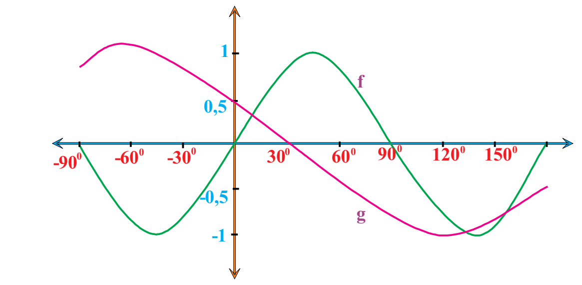
cos 2 3 $\sin^2$ 4 sin 2 0
1 2 $\sin^2$ 3 $\sin^2$ 4 sin 2 0
1 $\sin^2$ 4 sin 2 0
$\sin^2$ 4 sin 3 0
sin 1 sin 3 0
sin 1 or sin 3
90^0 k.360^0; k no solns.

Reminder: The sine anne functions have a range from -1 to 1. No. Solutions exist if values are out of this range

SOLVING EQUATIONS INVOLVING GRAPHS

The knowledge we have on general solutions is also applied when finding the points of intersection of two graphs.

Example : Below are shown the graphs of f={(x;y) : y = sin2x }and g(x) ={(x;y) : y= cos (x+60^0)}



Determine the values of x for which f(x) = g(x) for the domain x ϵ [-90^0 ; 180^0]

sin 2x cos x 60^0
sin 2x sin 90^0 x 60^0
sin 2x sin 30^0 x

1st quadrant		2nd quadrant
2x 30^0 x k.360^0	or	2x 180^0 30^0 x k.360^0
2x x 30^0 k.360^0		2x 150^0 x k.360^0
3x 30^0 k.360^0		2x 150^0 x k.360^0
x 10^0 k.120^0		2x x 150^0 k.360^0
		x 150^0 k.360^0; k

To obtain specific solutions we substitute integral values of k.

For k = 0 : x = 10° + (0).120° = 10°
For k = 0 : x = 150° + (0).360° = 150°
For k = 1 : x = 10° +(1).120° = 130°
For k = 1 : x = 150° +(1).360° = 510° * reject as out of domain
For k = -1 : x = 10° +(-1).120° = -110° * reject as out of domain
For k = -1 : x = 150° +(-1).360° = -210° * reject as out of domain

Solution Set = { 10° ; 130° ; 150°}