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MATHEMATICS
NUMBER PATTERNS

Co-ordinator: M. KHAN
Southlands Secondary

Key words:

A **sequence or progression** is an ordered list of terms of the form: $T_1; T_2; T_3; ....; T_n$ where T_n is the nth term or general term. T_1 is denoted by "a".

A **series** is formed when the terms are combined: $T_1 + T_2 + T_3 + T_4 + ...+T_n$

Some famous sequences are given below:

1. Write down the next three terms in the following consistent patterns:

a. 1; 1; 2; 3; 5; 8; 13; ....
The next two terms will be **21 and 34**. This is the famous Fibonacci sequence.

b. 2; 5; 7; 12; 19; 31; ....
The next two terms will be **50 and 81**. This is known as a Lucas sequence.

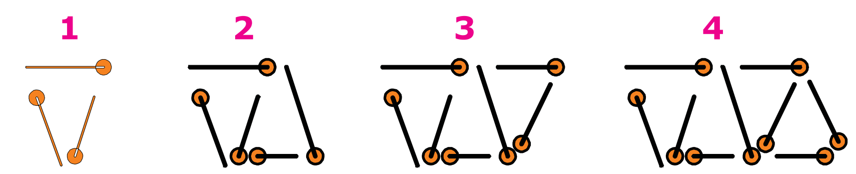
c. $\frac{1}{2}; \frac{1}{3}; \frac{1}{4}; ......$
The next two terms will be $\frac{1}{5}$ and $\frac{1}{6}$. Known as a Harmonic Sequence.

d. 1; 3; 7; 15; 31; 63; 127; 255; ....
This interesting pattern is 1 less than powers of 2. Known as a Mersenne sequence. The next two numbers will be **511 and 1023**.

e. 2; 3; 5; 7; 11; ....
These are the prime numbers. The next two terms will be **13 and 17**.

Number patterns can be illustrated by using pictures as shown in the following examples:

Diagram 1: Patterns made from matchsticks.



The number of match-sticks used to make each picture can be written as: 3; 5; 7;9; ....

n	1	2	3	n
T_n	3	5	7	

2 2

Obtain the first difference: T_2 T_1 =2 and T_3 T_2 = 2. Since the first difference is constant we know that the general term of this pattern will be linear and can be obtained by using the formula:

$T_n = a + (n - 1)d$
$T_n = 3 + (n - 1)2$
$= 3 + 2n - 2$
$= 2n + 1$

This formula is especially useful in obtaining the value of any term in this pattern. If we needed to know the number of match-sticks used to make the 20th arrangement we perform the following computation:

$T_n = 2n + 1$
$T_{20} = 2(20) + 1$
$= 41$

To obtain the total number of match-sticks used for 20 arrangements we use the following idea attributed to Wilhelm Gauss.
The series of terms : 3 ; 5 ; 7 ; 9 ; ....; 35; 37 ; 39 ; 41

Pairing the first term with the last term and the second with the second last and so on we notice that the sum of each pair is the same as (3 +41)=44. So 20 terms paired gives $\frac{20}{2}$.

Now the total number of match-sticks can be found.

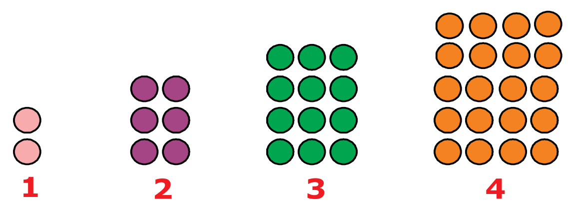
$S_{20} = \frac{20}{2}[3 + 41]$
$= 440$

This is the formula you might be used to seeing as: $S_n = \frac{n}{2}[2a + (n - 1)d]$

Diagram 2: Arrangement of beads.

In this pattern the second difference is constant. In this case the general term will be quadratic in form i.e $T_n = an^2 + bn + c$

From the sequence we can obtain the 5th term by adding 10 to 20 giving 30 and for the 6th term adding 12 to 30 giving 42.



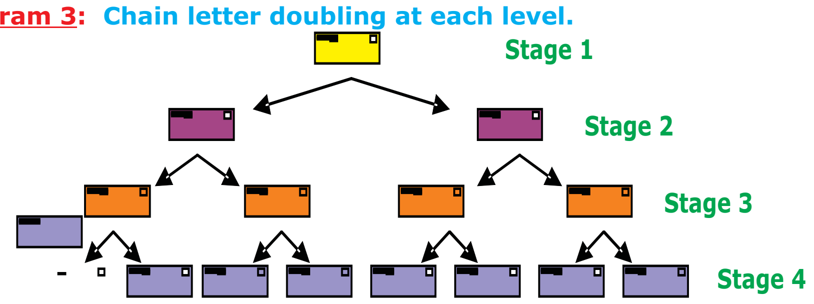
n	1	2	3	4
T_n	2	6	12	20

d_1: 4 6 8
d_2: 2 2

To obtain the general term we can use the following generalities:

$2a = d_2$; $3a + b = T_2 - T_1$; $a + b + c = T_1$
$2a = 2$
$\therefore a = 1$
sub a = 1 in $3a + b = T_2 - T_1$
$3 + b = 4$
$b = 1$
sub a = 1 and b = 1 in: $a + b + c = T_1$ we get:
$1 + 1 + c = 2$
$c = 0$
$\therefore T_n = 1n^2 + 1n + 0$
$= n^2 + n$

Diagram 3: Chain letter doubling at each level.



n	1	2	3	4
T_n	1	2	4	8

d_1: 1 2 4
d_2: 1 2

In this pattern we note that the 1st difference and the 2nd difference is not constant but there exists a constant ratio between successive terms of r= 2.

To find the general term we use the formula: $T_n = a.r^{n-1}$ to obtain,
$T_n = ar^{n-1}$
$= 1.2^{n-1}$
$= 2^{n-1}$

This allows us to find the number of letters in any stage. To find ,for example the number of letters in the 8th stage we substitute and obtain:
$T_n = 2^{n-1}$
$T_8 = 2^{8-1}$
$= 2^7 = 128$

To find the sum we use the formula: $S_n = \frac{a(r^n - 1)}{r - 1}$ allowing us to obtain: $S_n = \frac{1(2^n - 1)}{2 - 1} = \frac{2^n - 1}{1} = 2^n - 1$

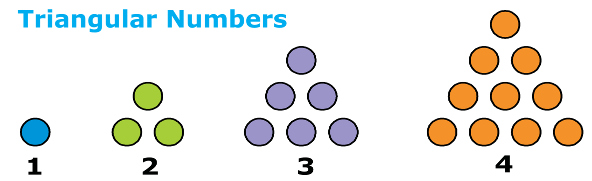
Now to find the sum of 10 terms we get: $S_{10} = 2^{10} - 1 = 1023.$

SIGMA NOTATION
This notation allows us to write the sum of the series in a shortened form.

$\sum_{k=1}^{n=10}(2k + 1)$ means the sum of first 20 terms of the series generated by the general term formula $T_k = 2k+1$. To evaluate it we generate specific terms by substituting k=1 ; k=2 ; k=3 and so on. The series is: 3 + 5 + 7+ ...Which is our first example.

$S = \frac{n}{2}[2a + (n - 1)d]$
Now $S_{20} = \frac{20}{2}[2(3) + 19(2)]$
$= 440$

Diagram 4: Triangular Numbers



In the above arrangement we get:

n	1	2	3	4
T_n	1	3	6	10

d_1: 2 3 4
d_2: 1 1

In this pattern the second difference is constant . The general term will be quadratic in form i.e $T_n = an^2 + bn + c$

From the sequence we can obtain the 5th term by adding 5 to 10 giving 15 and for the 6th term adding 6 to 15 giving 21.
To obtain the general term we use the following as illustrated:

$2a = d_2$; $3a + b = T_2 - T_1$; $a + b + c = T_1$
$2a = 1$
$\therefore a = \frac{1}{2}$
sub $a = \frac{1}{2}$ in $3a + b = T_2 - T_1$
$\frac{3}{2} + b = 2$
$b = \frac{1}{2}$
sub $a = \frac{1}{2}$ and $b = \frac{1}{2}$ in: $a + b + c = T_1$ we get:
$\frac{1}{2} + \frac{1}{2} + c = 1$
$c = 0$
$\therefore T_n = \frac{1}{2}n^2 + \frac{1}{2}n + 0$
$= \frac{1}{2}n^2 + \frac{1}{2}n$

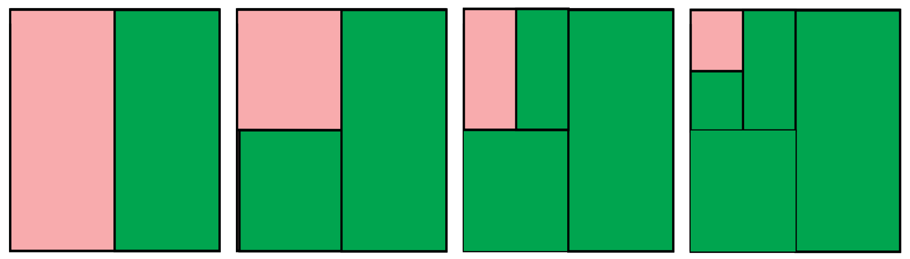
It is interesting to note that this is half of what we obtained in example 2. Known as Triangular Numbers.
The nth term formula allows us to find the value of a term in this pattern as well as finding the position of the term if the value is known.

91 is a Triangular Number. Determine its position in this pattern.

$\therefore T_n = \frac{1}{2}n^2 + \frac{1}{2}n = 91$
$n^2 + n = 182$
$n^2 + n - 182 = 0$
$n = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$
$= \frac{-1 + \sqrt{(1)^2 - 4(1)(-182)}}{2(1)}$
$= \frac{-1 + \sqrt{729}}{2}$
$n = 13$

To solve this quadratic equation each term was multiplied by 2 to get rid of the denominators. If factorization eludes us, use the Quadratic Formula to obtain the positive root of the equation.

Diagram 5: Converging Geometric Patterns
Example 1:



In the patterns above each consecutive pattern has more shaded rectangles than the previous one.
The area of the shaded portion of the first pattern is $\frac{1}{2}$ square units.

The Shaded area in Pattern 2 is $\frac{1}{2} + \frac{1}{4}$. Notice how each time ½ of the unshaded part now gets shaded.

Pattern 3 will have shaded area = $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ square units.

Pattern 4 will have shaded area = $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ square units.

Pattern n will have shaded area = $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{2^n}$

This can be written in Sigma notation as: $\sum_{i=1}^{n}\frac{1}{2^n}$

Evaluating $\sum_{i=1}^{3}\frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ and

Evaluating $\sum_{i=1}^{4}\frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$ and adding further terms

gives: $\sum_{i=1}^{5}\frac{1}{2^n} = \frac{31}{32}$ and $\sum_{n=1}^{6}\frac{1}{2^n} = \frac{63}{64}$

We notice that the sum is increasing but very minutely.
Evaluating:

$S_n = \frac{a(r^n - 1)}{r - 1}$

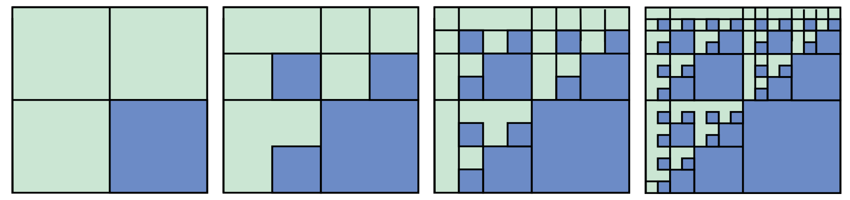
$S_{100} = \frac{\frac{1}{2}[(\frac{1}{2})^{100} - 1]}{\frac{1}{2} - 1}$
$= 1$

In fact as we add more and more terms the sum remains at 1. This series is said to have a limit of its sum i.e it converges to the value 1.
This occurs only when $-1 < r < 1$.
Note that this means that the terms of the geometric sequence have to be decreasing.

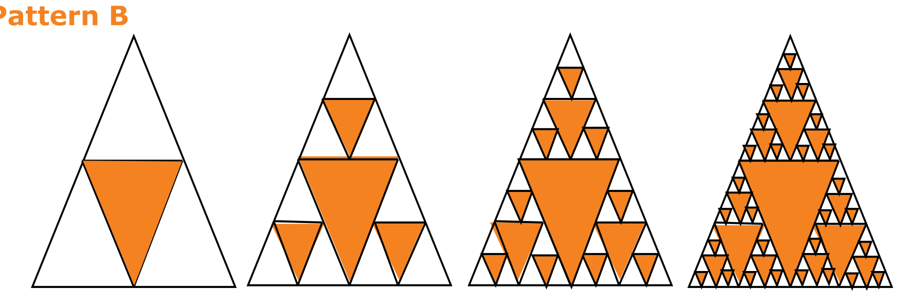
We can easily compute the value to which a decreasing geometric sequence converges by using the formula: $S_\infty = \frac{a}{1 - r}$

Applying the formula to the above problem gives: $S_\infty = \frac{\frac{1}{2}}{1 - \frac{1}{2}}$
$= 1$

Example 2:
Pattern A



Pattern B



The above patterns have the same consistent behaviour as illustrated below:

The area of the shaded portion of the first pattern is $\frac{1}{4}$ square units.

The shaded area in Pattern 2 is $\frac{1}{4} + \frac{3}{16}$. The area of the shaded portions of Patterns 3 is $\frac{1}{4} + \frac{3}{16} + \frac{9}{64}$ and in Pattern 4 $\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256}$.

Pattern n of each of the examples will have the shaded part as the following series: $\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256} + ... + \frac{3}{4^n}$.

$T_n = ar^{n-1}$
$= \frac{1}{4}.(\frac{3}{4})^{n-1}$
$= \frac{3^{n-1}}{4.4^{n-1}}$
$= \frac{3^{n-1}}{4^{1+n-1}}$
$= \frac{3^{n-1}}{4^n}$

In sigma notation this can be written as: $\sum_{i=1}^{n}\frac{3^{n-1}}{4^n}$

To calculate the area of the shaded part if this pattern continues without end , we evaluate the sum to infinity of the series which gives:

$S_\infty = \frac{a}{1 - r}$
$= \frac{\frac{1}{4}}{1 - \frac{3}{4}}$
$= 1$