MATHEMATICS

FINANCIAL MATHEMATICS

K12 SCHOOLS
YOUR EDUCATION PORTAL

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ANNUITIES (continued from Vol 6)

Annuity paid at the beginning of the interval.

Example 2. R12500 is deposited into an account annually for 6 years at a rate of 12% p. a compound interest. Calculate the future value of the investment after 6 years if deposits commence at the beginning of the year. To draw the time line:

- start at T_0 and end at T_6.
- put arrows where the cash flows occur.

Annuity due so : $i = 0,12$ *and* $n = 6$ *and* $x = 12500\ 1\ 0,12$

Substituting we obtain:

$$F_v = \frac{x\ 1\ i\ 1}{i}$$

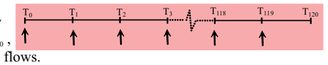
$$\frac{12500\ 1\ 0,12\ \ 1\ 0,12^{\ 6}\ 1}{0,12}$$

$$\frac{12500\ 1,12\ \ 1,12^{\ 6}\ 1}{0,12}$$

R113 612,65

Example 3. Determine the monthly contribution that a person needs to make if wants to have R 300 000 after 10 years if the bank pays 12% p.a compounded monthly and the contributions start immediately.

Firstly work out what the end T will be by performing this operation, $T_n = T_{10x12} = T_{120}$, we know the first is T_0. Illustrate the cash flows.



Remember you can also work out n = 119-0+1=120

This is an Annuity due.

$i = \frac{12}{1200} = 0,01$, $n = 10\ 12\ 120$, $F_v = 300\ 000$

$1\ i\ 1,01$, $x = x\ 1,01$

Substituting we obtain:

$$F_v = \frac{x\ 1\ i\ 1}{i}$$

$$300\,000 = \frac{x\ 1,01\ \ 1,01^{\ 120}\ 1}{0,01}$$

Isolating x we get:

$300\,000\ 0,01 = x\ 1,01\ \ 1,01^{\ 120}\ 1$

$3000 = 1,01^{\ 120}\ 1 \quad x\ 1,01$

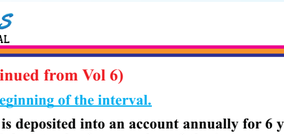
$1304,1284\ldots = 1,01 \quad x$

$x = R1291,22$

PRESENT VALUE OF AN ANNUITY

Example 1. A person wants to make 4 equal annual withdrawals from a fund that he will set up. The bank offers a rate of 10% per annum compound interest. What amount must he invest now if he wishes to make annual withdrawals of R 12000.

In this situation we assume that when he makes his fourth withdrawal there will be no more money in the account. The time line below illustrates the scene:



Here we need to work out the Present Value of each of the four withdrawals and find the sum of them.

P_v 1st withdrawal : $12000\ 1+0,1^{\ 1} = 12000\ 1,1^{\ 1}$

P_v 2nd withdrawal : $12000\ 1+0,1^{\ 2} = 12000\ 1,1^{\ 2}$

P_v 3rd withdrawal : $12000\ 1+0,1^{\ 3} = 12000\ 1,1^{\ 3}$

P_v 4th withdrawal : $12000\ 1+0,1^{\ 4} = 12000\ 1,1^{\ 4}$

The Total Present value of these regular withdrawals can be easily calculated by setting up the sum of their individual present values. The series is shown below:

$P_v \quad 12000\ 1,1^{\ 1} \quad 12000\ 1,1^{\ 2} \quad 12000\ 1,1^{\ 3} \quad 12000\ 1,1^{\ 4}$

R38 038,39

Notice that the above series forms a geometric sequence with a = $12000\ 1,1^{\ 1}$ and r = $1,1^{\ 1}$ and n = 4.

So we can use the formula $S_n = \frac{a\ 1\ r^n}{1\ r}$ to calculate the sum of the series.

Now using S_n

$$\frac{12000\ 1,1^{\ 1}\ 1\ \ 1,1^{\ 4}}{1\ \ 1,1^{\ 1}}$$

R38 038,39

The easiest way of course is to use the Present Value formula which is:

$$P_v = \frac{x\ 1\ \ 1\ i^{\ n}}{i}$$

We simply substitute the values below:

$i = 0,1$; $1\ i\ 1,1$; $n = 4$; $x = 12000$

$$P_v = \frac{12000\ 1\ \ 1\ \ 0,1^{\ 4}}{0,1}$$

$$\frac{12000\ 1\ \ 1,1^{\ 4}}{0,1}$$

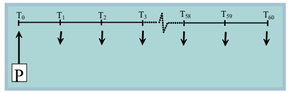
R38 038,39

Example 2. Determine the amount of money that needs to be invested now to obtain annuity payments of R3500 per month for 5 years if interest is 7% p.a compounded monthly.

To assist we will draw the time line showing the cash flows.

T_0 is where our investment will occur. The other arrows show where the intended withdrawals will take place. Don't forget that money available presently is worth more than the money available in the future.

For 5 years there will be 5 x 12 = 60 monthly withdrawals.



$x = 3500$ $n = 5\ 12\ 60$ $i = \frac{7}{1200}$

$$P_v = \frac{x\ 1\ \ 1\ i^{\ n}}{i}$$

$$\frac{3500\ 1\ \ 1\ \ \frac{7}{1200}^{\ 60}}{\frac{7}{1200}}$$

R176 756 ,98

What this means is that for R 176 756,98 investment now you will be able to withdraw 60 x 3500 = R210 000. The difference of R33 243 ,02 represents the interest earned on your initial investment.

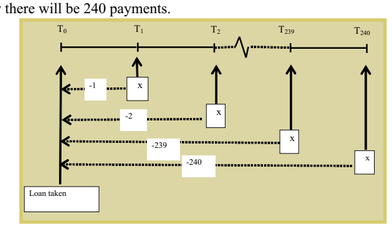
BOND AND LOAN REPAYMENTS

When dealing with Bond and Loan Repayments we consider the loan amount to be our present value and work out our repayments directly from the Present Value Formula: $P_v = \frac{x\ 1\ \ 1\ i^{\ n}}{i}$

Example 1. A new complex has 1 ½ bedroomed units for sale at R780 000. A lady purchases the unit as a starter home and takes a 100% bond with the building society at 14,76% per annum compounded monthly.

1.1 Determine her monthly repayments if her bond is paid over 20 years.

A time line is drawn to illustrate the situation. Note that payments start one month after the loan is drawn. Clearly there will be 240 payments.



To work this out we use the fact that the sum of the present values of each payment with interest must equal the amount of the loan drawn plus its interest.

$x = ?$, $n = 240$, $i = \frac{14,76}{1200} = 0,0123$, $P_v = 780\,000$

$$P_v = \frac{x\ 1\ \ 1\ i^{\ n}}{i}$$

$$780000 = \frac{x\ 1\ \ 1\ \ 0,0123^{\ 240}}{0,0123}$$

$780000\ 0,0123 = x\ 1\ \ 1,0123^{\ 240}$

$$x = \frac{9594}{1\ \ 1,0123^{\ 240}}$$

R10 132,91

1.2 Suppose she pays a deposit of 20% when buying the unit ,determine her monthly instalment if the rate and payment interval remains the same.

The value of the loan amount needs to be worked out first.

$x = ?$, $n = 240$, $i = 0,0123$

$P_v \quad 780\,000 \quad 20\%\ of\ 780000 = 624000$

Substituting the values we obtain:

$$P_v = \frac{x\ 1\ \ 1\ i^{\ n}}{i}$$

$$624000 = \frac{x\ 1\ \ 1\ \ 0,0123^{\ 240}}{0,0123}$$

$624000\ 0,0123 = x\ 1\ \ 1,0123^{\ 240}$

$$x = \frac{7675,20}{1\ \ 1,0123^{\ 240}}$$

R8 106,33

Example 2. A loan of R 260 000 is obtained at 18 % compounded monthly for 25 years.

2.1 Calculate the monthly bond repayment.

$x = ?$, $n = 25\ 12 = 300$, $i = \frac{18}{1200} = 0,015$, $P_v = 260000$

$$P_v = \frac{x\ 1\ \ 1\ i^{\ n}}{i}$$

$$260000 = \frac{x\ 1\ \ 1\ \ 0,015^{\ 300}}{0,015}$$

$260000\ 0,015 = x\ 1\ \ 1,015^{\ 300}$

$$x = \frac{3900}{1\ \ 1,015^{\ 300}}$$

R3 945,32

2.2 Calculate how much of the payments is only interest.

Interest = monthly instalment 300 Loan amount

= 3945,32 300 260000

R923 595,36

LOAN BALANCES

To work out a loan balance we calculate the sum of the Present values of the remaining payments i.e we use the present value formula.

Example 1: Mr. Khumalo buys a car for R135 000. He pays with a deposit of R 25 000 and finances the balance with a bank loan. The interest charged is 14,4 % p. a compounded monthly.

1.1 Calculate his monthly payments if he pays it over 42 months.

$x = ?$, $n = 42$, $i = \frac{14,4}{1200} = 0,012$

$P_v \quad 135000\ \ 25000 = 110000$

$$P_v = \frac{x\ 1\ \ 1\ i^{\ n}}{i}$$

$$110000 = \frac{x\ 1\ \ 1\ \ 0,012^{\ 42}}{0,012}$$

$110000\ 0,012 = x\ 1\ \ 1,012^{\ 42}$

$$x = \frac{1320}{1\ \ 1,012^{\ 42}}$$

R3349,61

1.2 Determine his balance immediately after his 30th instalment.

Clearly there are 12 payments of R3349,61 left. We now use the present value of these Payments over 12 months.

$x = 3349,61$, $n = 42\ 30\ 12$, $i = 0,012$, $P_v = ?$

$$P_v = \frac{x\ 1\ \ 1\ i^{\ n}}{i}$$

$$\frac{3349,61\ 1\ \ 1,012^{\ 12}}{0,012}$$

R37 228,05

1.3 Mr Khumalo decides to pay R3800 each month from the 1st instalment. Calculate how Long he will take to pay off the loan.

$x = 3800$, $n = ?$, $i = 0,012$

$P_v \quad 110000$

$$P_v = \frac{x\ 1\ \ 1\ i^{\ n}}{i}$$

$$110000 = \frac{3800\ 1\ \ 1\ \ 0,012^{\ n}}{0,012}$$

$110000\ 0,012 = 3800\ 1\ \ 1,012^{\ n}$

$$\frac{1320}{3800} = 1\ \ 1,012^{\ n}$$

$$1,012^{\ n}\ \ 1 = \frac{1320}{3800}$$

$1,012^{\ n} = 0,65263$

$n \log 1,012 = \log 0,65263$

$$n = \frac{\log 0,65263}{\log 1,012}$$

$n = 35,775$

He will pay off the loan in 36 months.

Isolating the term with *n* on right hand side.

1.4 Determine the amount of his final payment.

To determine the amount of his final payment we need to first work out the present value of his payments Over 35 months.

$x = 3800$, $n = 35$, $i = 0,012$, $P_v = ?$

$$P_v = \frac{x\ 1\ \ 1\ i^{\ n}}{i}$$

$$\frac{3800\ 1\ \ 1,012^{\ 35}}{0,012}$$

R108 081,02

Amount outstanding at the beginning of 36th month = 110 000-108 081,02 = R 1918 ,98

Amount of final payment $= P(1\ \ i)^n$

$1918,98\ 1\ \ 0,012^{\ 36}$

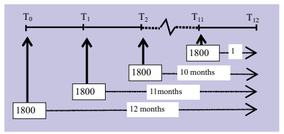
2948,28

SINKING FUNDS

A sinking fund is a fund set up to offset the cost that will be incurred when a purchase is made in the future. We will use our formulae for Annuities to solve these questions.

Example 1. A school sets up a sinking fund at the beginning of the month to replace its duplicating machine in 1 year's time. The school can afford an amount of R1800 each month. They intend to make 12 deposits starting immediately. Calculate the value of the fund after one year if interest is calculated at 14 % per annum.

We need to find the sum of the future values of each monthly deposit. A time line is drawn to show when the deposits start and end. This is important as we have seen that the Future value depends on the accumulation factor of the last deposit.



Clearly the 12 deposits end at T_{11} , so we cater for the accumulation factor for that deposit in our calculations.

$i = \frac{14}{1200}$; $n = 12$; $x = 1800\ 1\ \frac{14}{1200}$ Substituting we obtain:

$$F_v = \frac{x\ 1\ \ i^{\ n}\ 1}{i}$$

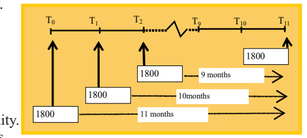
$$1800\ 1\ \ \frac{14}{1200}\ \ 1\ \ \frac{14}{1200}^{\ 12}\ \ 1$$

$$\frac{14}{1200}$$

R23 310,16

Example 2. In this example we are going to use the situation of the previous question about the school but we are going to phrase it differently.

A school sets up a sinking fund to replace its duplicating machine in 1 year's time. The school can afford an amount of R1800 each month starting immediately. Calculate the value of the fund immediately after the 12th deposit is made if interest is calculated at 14 % per annum. This is an ordinary annuity. A time line is drawn to show when the deposits start and end.



We will calculate the future value of the sinking fund using the Future value formula as this is an ordinary annuity.

$$F_v = 1800\ \ 1800\ 1\ \ \frac{14}{1200}^{\ 1}\ \ 1800\ 1\ \ \frac{14}{1200}^{\ 2}\ \ \ldots\ \ 1800\ 1\ \ \frac{14}{1200}^{\ 11}$$

$$x = 1800\ ;\ n = 12\ ;\ i = \frac{14}{1200}$$

$$F_v = \frac{x\ 1\ \ i^{\ n}\ 1}{i}$$

$$1800\ 1\ \ \frac{14}{1200}^{\ 12}\ \ 1$$

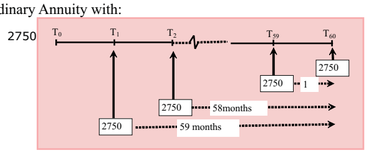
$$\frac{14}{1200}$$

R23 041,34

Example 3. A company establishes a sinking fund to replace its delivery truck in 5 years time. It deposits R2750 monthly into the fund starting the end of the month. Calculate the value of a replacement truck that they could purchase with the available funds immediately after the last deposit is made if interest is calculated at 13 % per annum and their old delivery truck is traded in for R 28 500,00.

The time line shows that this is an Ordinary Annuity with:

n=12 5=60 ; $i = \frac{13}{1200}$; $x = 2750$



Substituting we obtain the future value of the sinking fund

$$F_v = \frac{x\ 1\ \ i\ \ 1}{i}$$

$$\frac{2750\ 1\ \ \frac{13}{1200}^{\ 60}\ \ 1}{\frac{13}{1200}}$$

R230 709,74

To obtain the total available funds we need to add the trade in value of the old truck giving:

R230 709,74 + R28 500 = R 259 209 ,74

Example 4. (Adapted from Grade 12 Exemplar)

A new truck costs R1, 25 million. The value of the truck will depreciate on a reducing balance per annum to R554 631 ,64 in 5 years time. The price of a new truck rises by 10 % p.a.

4.1 Calculate the rate of depreciation of the truck.

Using $A = P\ 1\ \ i^{\ n}$ we get :

$554\,631,64 = 1\,250\,000\ 1-\frac{r}{100}^{\ 5}$

$$\frac{554\,631,64}{1\,250\,000} = 1-\frac{r}{100}^{\ 5}$$

$$\sqrt[5]{\frac{554\,631,64}{1\,250\,000}} = 1\ \ \frac{r}{100}$$

$$\frac{r}{100} = 1\ \ \sqrt[5]{\frac{554\,631,64}{1\,250\,000}}$$

$$i = 100\ 1\ \ \sqrt[5]{\frac{554\,631,64}{1\,250\,000}}$$

15%

4.2 Calculate the value of the sinking fund that needs to be set up to pay for the new truck, if the old truck is traded in.

In 5 years a new truck will cost :

$A = P\ 1\ \ i^{\ n}$

$1\,250\,000\ 1\ \ \frac{10}{100}^{\ 5}$

R 2 013 137 ,50

The value of the sinking fund :

= Cost of new truck - trade in value of old truck

=2 013 137,50 -554 631,64

=R 1 458 505 ,86

4.3 Calculate the percentage increase in the cost of a new truck.

$$\%\ increase = \frac{price\ increase}{old\ price}\ \ \frac{100}{1}$$

$$\frac{2013137,50\ \ 1250000}{1250000}\ \ \frac{100}{1}$$

61,05%